THE SILVERMAN-TOEPLITZ THEOREM

by

BRIAN RUDER

B. A., Fort Hays Kansas State College, 1964

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1966

Approved by:

Charles F. Koch
Major Professor

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INTRODUCTION

The modern theory of summable series has its origin in the early eighteenth century. In 1713, a letter written by Leibniz to Christian Wolf discussing the series

(1) $1 - 1 + 1 - 1 + \ldots,$

was published in the Acta Eruditorum. Leibnitz tried to justify the value $\frac{1}{2}$ for the series. He felt that the value $\frac{1}{2}$ for the series was reasonable on the basis of his "law of continuity" and on the basis of the relation:

(2) $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots,$$

which is true for values of x numerically less than unity. However, Leibnitz felt that the value $\frac{1}{2}$ should be obtainable from the series (1) without recourse to the series (2). One of his arguments for the value $\frac{1}{2}$ was the following: If we take the sum of an even number of terms of the series (1), the value is always zero; if we take the sum of an odd number of terms, the value is always unity. When we pass to the case of an infinite number of terms, there is no reason to consider that we have either an odd or an even number of terms, and therefore no reason for assigning either the value unity or zero to the series, but rather it is reasonable to assign some intermediate value. Moreover, in the process of allowing the number of terms to become infinite, the values zero and unity for the series occur with equal frequency. He therefore considered it justifiable, on the basis of probability, to assign to the infinite series a value intermediate between unity and zero, which is precisely their arithmetic mean, namely $\frac{1}{2}$.

After this publication of Leibnitz, Euler started work in this area, and is credited with formalizing the theory of divergent series and summability methods. Since that time, many prominent mathematicians have taken an interest in the subject and have developed the theory to what it is at the present time.

Prior to a discussion of divergent series, summability methods, and the Silverman-Toeplitz Theorem, it is essential that a few definitions be introduced as a means of communications. In passing, one notes that the theory of convergent series is contained in the theory of divergent series in that convergence is nothing more than a logically simple method of assigning values to infinite series. The fact that there is more literature written on convergence than divergence is quite logical since there have been more elementary applications of convergent series. However this does not in any way detract from the interest and usefulness of the theory of divergent series.

Definition 1

An infinite series $\sum_{n=0}^{\infty} a_n$ is a symbol for a definite sequence of complex numbers deducible from it, namely the sequence of partial sums. The series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

has n th partial sums (denoted by S_n) given by

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$$

Definition 2

An infinite series $\sum_{n=0}^{\infty} a_n$ with an n th partial sum S_n, is said to be convergent, if $\lim_{n\to\infty} S_n = S$, and we call S the "sum" of the series.

Definition 3

An infinite series $\sum_{n=0}^{\infty} a_n$ with n th partial sum S_n, is said to diverge definitely to $+\infty$ if $\lim_{n\to\infty} S_n = +\infty$, and is said to diverge definitely to $-\infty$ if $\lim_{n\to\infty} S_n = -\infty$.

Definition 4

An infinite series $\sum_{n=0}^{\infty} a_n$ with n th partial sum S_n is said to diverge indefinitely if it is neither convergent nor definitely divergent. The series (1), 1 -1 +1 - ..., is an example of an indefinitely divergent series.

In the study of summability, the only divergent series of any interest, and the only ones that will be considered in this paper will be those which are indefinitely divergent.

Definition 5

A summability method is a process which attempts to assign a number, complex or real, to infinite series.

At this point, a few properties which are desirable in a summability method will be discussed. The first property which the method should possess is non-contradiction to the idea of convergence. (One wants the method to have this property, because convergent series are of such great value, any summability method which contradicts it, could scarcely be expected to have much value.) Therefore one will want the method to satisfy a permanence condition.

Definition 6

If an infinite series $\sum_{n=0}^{\infty} a_n$ converges to S, and if V-$(\sum_{n=0}^{\infty} a_n) = S_1$, where V is a summability method, we say that the summability method satisfies the permanence condition. The second property that the method should possess is closely related to the permanence property and is called consistence.

Definition 7

If for every infinite series $\sum_{n=0}^{\infty} a_n$ which converges to S, we have $V(\sum_{n=0}^{\infty} a_n) = S$, where V is a summability method, we say that the summability method V satisfies the consistency property. If a summability method satisfies both the permanence property and the consistency property, the following definition is applicable.

Definition 8

A summability method is said to be regular if it satisfies the permanency property and it also satisfies the consistency property.

There are other requirements that one might want a summability method to possess, such as an extension property, (which means that the summability method sums at least one divergent series) or a compatibility condition. (This property guarantees that if an infinite series $\sum_{n=0}^{\infty} a_n$ is summable by two different methods, that both methods yield the same value.) Most methods do satisfy these properties, but in this paper only the property of regularity will be discussed, so no other properties will be dealt with in detail.

As should be quite evident from the definition of a regular summability method, any summability method that is not regular is of very little practical value. Since regularity is an important property, one would like to be able to determine whether an arbitrary summability method is regular or not. With the aid of the Silverman-Toeplitz Theorem, one is able to give a partial solution to this problem. If one is able to put a given summability method into a form in which the Silverman-Toeplitz Theorem can be applied, then it is possible to determine whether the method is regular or not. Many summability methods can be put into this form, so the value of this theorem is evident.

In this paper, the Silverman-Toeplitz Theorem will be proved and some of the applications of the Theorem will be given.

Before discussing the Theorem, two definitions and examples of well known summability methods will be given which shall be referred to throughout the rest of this paper.

CESARO SUMMABILITY

Given the infinite series $\sum_{n=0}^{\infty} u_n$, Let $S_n = \sum_{k=0}^{n} u_k$.

Set $C_n = S_0 + S_1 + S_2 + \dots + S_n$.

Then one says that the given series is summable (C,1) to the sum C if

$$\lim_{n \to \infty} \frac{C_n}{n+1} = C.$$

This method of summability is due to Ernesto Cesaro. A more general Cesaro summability method is (C,K) summability, defined in the following manner:

$$S_n^{(0)} = S_n \text{ , and for } k \geq 1,$$

$$S_n^{(k)} = S_0^{(k-1)} + S_1^{(k-1)} + \dots + S_n^{(k-1)};$$

$$(n = 1,2,3,\dots)$$

and one now considers

$$C_n^{(k)} = \frac{S_n^{(k)}}{\binom{n+k}{k}} \quad \text{for each fixed } k.$$

If for some value of k, $C_n^{(k)} \longrightarrow C$, one says the sequence $\{S_n\}$ is summable (C,K) to the sum C.

In this paper, only (C,1) summability will be considered as it lends itself readily to examples.

As an example of the application of the Cesaro method, note that, with this method of summability, the series (1) is summable to the value $\frac{1}{2}$, which is consistent with Leibnitz's result.

As a second example consider the problem of finding the analytic continuation of the infinite series $\sum_{k=1}^{\infty} z^k$ which converges to $\frac{1}{1-z}$ for $|z| < 1$. For this series, $S_n = \frac{1-z^{n+1}}{1-z}$ and

$$C_n = S_0 + S_1 + S_2 + \ldots S_n = \frac{(1-z)^0 + (1-z)^1 + \ldots + (1-z)^n}{1-z}$$

$$C_n = \frac{(1-z)(n+1 + nz + (n-1)z + \ldots + (n-(n-1))z}{1-z} .$$

In considering the limit of $\frac{C_n}{n+1}$, one finds that the limit exists if and only if $|z| \leq 1$; $z \neq 1$. To verify the case where $z = -1$, consider the partial sums.

$$S_0 = 1,\ S_1 = 0,\ S_2 = 1,\ \ldots ,$$

$$C_n = S_0 + S_1 + S_2 + \ldots + S_n, \text{ where } S_n = \tfrac{1}{2}\,(1 + (-1)^n).$$

Therefore, $\frac{C_n}{n+1} = \frac{(n+1) + \frac{1}{2}(1 + (-1)^n)}{2(n+1)} = \frac{1}{2} + \frac{1 + (-1)^n}{4(n+1)}$ and

$\lim_{n \to \infty} \frac{C_n}{n+1} = \frac{1}{2}$. It should be noted that this value of $\frac{1}{2}$ is in accordance with what one would expect to obtain if -1 were substituted for x in the term $\frac{1}{1-x}$, to which the original series converged.

Before introducing the second summability method, the relation between infinite series and their corresponding sequences will be discussed in a little more detail.

As was mentioned earlier in this paper, to every series there corresponds a sequence of partial sums. It is also true that to every infinite sequence there corresponds an infinite series. The limit of the sequence is the same as the sum of the infinite series, when either value exists.

Thus, in talking about summability methods, one is justified in operating with the sequence of partial sums. The relation of a series to its sequence was introduced above, so at this point, only the relation of a sequence to its series will be mentioned. Given a sequence of complex numbers, $(z_0, z_1, \ldots)$ one can write this as a series by letting $a_0, = z_0$, $a_1 = z_1 - z_0$, $a_2 = z_2 - z_1$, ... $a_n = z_n - z_{n-1}, \ldots$. One then has the series

$$\sum_{n=0}^{\infty} a_n = z_0 + \sum_{k=1}^{\infty} (z_k - z_{k-1}).$$

The reason for introducing the relation of series to sequence is because the Silverman-Toeplitz Theorem will be proved in the form where one considers a summability method as an operator on sequences rather than an operator on infinite series. In this case one can consider a summability method as a transformation, which takes the given set of partial sums into a new set of partial sums. At this point the second method of summability, due to Euler, may be introduced.

An infinite series $u_0 + u_1 + \dots$ and its sequence S_0, $S_1, \dots$ of partial sums, are said to be summable to t by the Euler transformation (or method of summability) E(r) of order r, r being a complex constant, if

$$t_n \longrightarrow t \quad \text{as} \quad n \longrightarrow \infty \quad \text{where}$$

$$t_n = t_n(r) = \sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} S_k .$$

Consider this method when applied to the series $\sum_{k=0}^{\infty} z^k$. From the definition of the transformation, one sees that the new sequence, in this case, is given by:

$$t_n = \sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} \cdot \frac{1-z^{k+1}}{1-z} \quad \text{where}$$

$$\frac{1-z^{k+1}}{1-z} = S_k = \sum_{n=0}^{k} z^n.$$

One can write this transformation as

$$\frac{1}{1-z} \left[\sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} - z \sum_{k=0}^{n} \binom{n}{k} (rz)^k (1-r)^{n-k} \right].$$

Now note that $\sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} = (r + 1-r)^n$, and that $\sum_{k=0}^{n} \binom{n}{k}$-$(rz)^k (1-r)^{n-k} = (rz + (1-r))^n$, and one can write the transformation as

$$\frac{1}{1-z}\ \left(1-z\ (rz + 1-r)^n\right)\ .$$

If $z \neq 1$, then the series is summable to its analytic continuation if and only if $|1-r + rz| < 1$; this is the same as

$$C(r) = \left|z + \frac{1-r}{r}\right| < \frac{1}{|r|}\ .$$

If $r \neq 0$ and r is fixed, the set of values of z for which the series is summable $E(r)$ consists of the interior of the circle $C(r)$ with center at the point $\frac{1-r}{r}$ and radius $\frac{1}{|r|}$. As a particular case consider $r = \frac{1}{15}$, then one has $C(\frac{1}{15}) = |z + 14| < 15$ for which the sequence is summable and this includes, and is larger than the original circle of convergence.

From these two examples, one can note that the Euler method is more powerful than (C,1) summability in the case of the geometric series. However, (C,1) summability is more powerful than the Euler method when one deals with certain other series.

Up to this point, nothing has been said about the regularity of these two methods. It will be shown (by making use of the Silverman-Toeplitz Theorem) that (C,1) summability is regular, and that E(r) is regular if and only if certain restrictions are placed on r.

The following infinite matrices will be used as reference matrices throughout the rest of this paper.

$$U \cdot S = \begin{Vmatrix} a_{00} & a_{01} & a_{02} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{n0} & a_{n1} & a_{n2} & \cdots \\ \cdot & & & \\ \cdot & & & \\ \cdot & \cdots & & \end{Vmatrix} \begin{Vmatrix} S_0 \\ S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ S_n \\ \cdot \\ \cdot \\ \cdot \end{Vmatrix} = \begin{Vmatrix} T_0 \\ T_1 \\ T_2 \\ \cdot \\ \cdot \\ \cdot \\ T_n \\ \cdot \\ \cdot \\ \cdot \end{Vmatrix} = T$$

U is an infinite matrix denoted by (a_{nv}). S and T are infinite column matrices. The product $U \cdot S = T$ expresses a method of transforming the sequence $\{S_n\}$ to the sequence $\{T_n\}$. In the problems to be discussed, the S matrix will have as elements the partial sums of the infinite series under consideration.

The method of proof selected for this paper requires that a few lemmas be proven prior to consideration of the Silverman-Toeplitz Theorem.

Lemma 1

If the series $\sum_{n=0}^{\infty} p_n q_n$ is convergent for every bounded or merely for every convergent sequence $\{p_n\}$, then the series $\sum_{n=0}^{\infty} q_n$ is absolutely convergent. It is convenient, at this place to introduce the function signum, abbreviated sgn, which is defined as follows:

$$\operatorname{sgn} z = \begin{cases} \dfrac{|z|}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 . \end{cases}$$

Proof.

If the sequence $\{p_n\}$ is required to be bounded, then let $p_n = \operatorname{sgn} q_n$ (which is obviously bounded) and one has

$$\sum_{n=0}^{\infty} p_n q_n = \sum_{n=0}^{\infty} \frac{|q_n|}{q_n} q_n = \sum_{n=0}^{\infty} |q_n| ,$$

which is absolutely convergent and the first part of the lemma is proved. If the sequence $\{p_n\}$ is convergent, the following argument can be given. Assume $\sum_{n=0}^{\infty} |q_n|$ is divergent. Let

$Q_n = \sum_{v=0}^{n} |q_v|$, then as $n \longrightarrow \infty$ Q_n approaches infinity monotonically. The series $\sum_{n=0}^{\infty} \frac{q_n}{Q_n}$ is divergent, for by writing

$$\sum_{v=n}^{n+s} \frac{|q_v|}{Q_v} = \sum_{v=n}^{n+s} \frac{Q_v - Q_{v-1}}{Q_v}$$

one notes that

$$\sum_{v=n}^{n+s} \frac{Q_v - Q_{v-1}}{Q_v} \geq \frac{Q_{n+s} - Q_{n-1}}{Q_{n+s}} = 1 - \frac{Q_{n-1}}{Q_{n+s}} \geq \tfrac{1}{2}$$

for s sufficiently large. Now set $p_n = \frac{|q_n|}{q_n Q_n}$ and it is true $p_n \to 0$ as n gets large since Q_n approaches infinity. Now by hypothesis $\sum_{n=0}^{\infty} p_n q_n$ converges, but

$$\sum_{n=0}^{\infty} p_n q_n = \sum_{n=0}^{\infty} \frac{|q_n|}{q_n Q_n} \cdot q_n = \sum_{n=0}^{\infty} \frac{|q_n|}{Q_n} ,$$

which leads to a contradiction and therefore our assumption must have been wrong and $\sum_{n=0}^{\infty} |q_n|$ is convergent and the lemma is proved. Lemma 1 shows that any condition on the U matrix given above which insures the existence of the T matrix for all bounded or convergent sequences $\{S_n\}$, will have to include the condition: the sum of the absolute values of the elements of each row of U is finite, that is

$$W_n = \sum_{v=0}^{\infty} |a_{nv}|$$

exists for each n. As a matter of convenience, let $V_n = \sum_{v=0}^{\infty} a_{nv}$ and $W_n = \sum_{v=0}^{\infty} |a_{nv}|$ for the rest of this paper.

Lemma 2

A necessary and sufficient condition that the sequence $\{T_n\}$ be bounded whenever the sequence $\{S_n\}$ is bounded is that the sequence $\{W_n\}$ be bounded.

Proof.

Consider the sufficiency first (that is if $\{S_n\}$ is bounded and $\{W_n\}$ is bounded, then $\{T_n\}$ is bounded.) Let M be the least upper bound (abbreviated l.u.b.) of $|S_n|$, and let N be the l.u.b. of W_n, then $|T_n| \leq M\cdot N$ and $\{T_n\}$ is bounded.

For the necessity of the condition, note first of all that the elements of each column of U must be bounded. To show that this is true consider the set of sequences $S_n = \delta_{nv}$, $n = 0,1,2, \ldots$ where δ_{nv} is the Kronecker delta defined as follows:

$$\delta_{nv} = \begin{cases} 1 & \text{if } n = v \\ 0 & \text{otherwise.} \end{cases}$$

Then one has $T_n = a_{nv}$, $n = 0,1,2,\cdots$ and this sequence must be bounded for each v according to the hypothesis of the lemma.

Now if W_n is unbounded, a bounded sequence $\{S_n\}$ will be constructed for which $\{T_n\}$ is unbounded. Without loss of generality, it may be assumed that W_n tends monotonically to infinity, for is this were not true, a subsequence $\{X_n\}$ of W_n for which this were true could be selected. Now define the sets of integers (u_n), (k_n), (y_n) as follows: $u_0 = 0$; $k_0 = 0$; y_0 is the smallest integer greater than $k_0 + 1$ such that $\sum_{v=y_0}^{\infty} |a_{0v}| \leq 1$.

The reason that one can do this follows from lemma 1, that is if the T matrix is to exist (and this is the only case to be considered here then $\sum_{v=0}^{\infty} |a_{nv}|$ must exist for all n and in particular for n = 0. If $\sum_{v=0}^{\infty} |a_{nv}|$ exists, then for v large enough, it is true that $\sum_{v=y_0}^{\infty} |a_{0v}| \leq 1$. Now suppose that the first m of each of these sets, (u_n), (k_n), and (y_n) have been chosen; choose u_m, k_m, y_m, as follows. First choose u_m as the smallest integer greater than u_{m-1} for which there are numbers k, greater than y_{m-1} such that

$$\text{(a)} \quad \sum_{v=0}^{k} |a_{uv}| \leq \frac{1}{3} W_u - 1 \qquad \text{for } u = u_m .$$

Now u_m exists since W_n tends monotonically to infinity and the elements of each column of U are bounded, i.e. $|a_{nv}| \leq M(v)$, $n = 0,1,2,\ldots$, $v = 0,1,2,\ldots$. This means that once u_{m-1} has been chosen, one can select v large enough so that a u_m exists which satisfies (a) because the columns of U are bounded. Let $k_m = y_{m-1} + 1$; k_m is then a k for which (a) is true. Take y_m as the smallest integer greater than $k_m + 1$ such that

$$\text{(b)} \quad \sum_{v=y_m}^{\infty} |a_{uv}| \leq 1 \qquad \text{for } u = u_m .$$

This is possible by lemma 1 as was shown above. The sets (u_n), (k_n), and (y_n) are completely defined and one has

$$k_0 < y_0 < k_1 < y_1 < k_2 < y_2 < k_3 < y_3 < \ldots .$$

Now define the bounded sequence $\{S_n\}$ for which $\{T_n\}$ will be unbounded, as

$$S_v = \begin{cases} \operatorname{sgn} a_{uv} & \text{if } k_n < v < y_n \qquad u = u_m \\ 0 & \text{otherwise} \qquad (y_{n-1} \leq v \leq k_n). \end{cases}$$

The sequence $\{S_n\}$ is obviously bounded and one can see for $u = u_n$

$$|T_u| = \left|\sum_{v=0}^{\infty} a_{uv} S_v\right| = \left|\sum_{v=0}^{\infty} a_{uv} \frac{|a_{uv}|}{a_{uv}}\right| = \sum_{v=0}^{\infty} |a_{uv}|.$$

Now

$$(1) \qquad \sum_{v=0}^{\infty} |a_{uv}| = \sum_{v=k_{n+1}}^{y_{n-1}} |a_{uv}| + R_u ,$$

where $\quad |R_u| < \sum_{v=0}^{k_n} |a_{uv}| + \sum_{v=y_n}^{\infty} |a_{uv}|.$

In defining $\{S_n\}$ many zeros were introduced and this is the reason for the strict inequality. Now to proceed, one has

$$W_u = \sum_{v=0}^{k_n} |a_{uv}| + \sum_{v=k_{n+1}}^{y_{n-1}} |a_{uv}| + \sum_{v=y_n}^{\infty} |a_{uv}| , \quad \text{and}$$

$$(2) \qquad \sum_{v=k_{n+1}}^{y_{n-1}} |a_{uv}| = W_u - \sum_{v=0}^{k_n} |a_{uv}| - \sum_{v=y_n}^{\infty} |a_{uv}| .$$

Note that R is positive so that $R > -R$. Now replace R by $-R$ in (1) and then use (2) to get

$$\left| T_u \right| = \sum_{v=0}^{\infty} \left| a_{uv} \right| = \sum_{v=k+1}^{y_{n-1}} \left| a_{uv} \right| + R_u ,$$

$$\geq W_u - 2 \sum_{v=0}^{k_n} \left| a_{uv} \right| - 2 \sum_{v=y_n}^{\infty} \left| a_{uv} \right| ,$$

$$\geq W_u - \left\{ \frac{2}{3} W_u + 2 - 2 \right\} = \frac{1}{3} W_u ,$$

and W_u tends monotonically to infinity. The last inequality was obtained by using (a) and (b), that is

$$- 2 \sum_{v=0}^{k_n} \left| a_{uv} \right| \leq - 2 \left(\frac{1}{3} W_u - 1 \right) = - \frac{2}{3} W_u + 2, \text{ and}$$

$$- 2 \sum_{v=y_n}^{\infty} \left| a_{uv} \right| \leq - 2 \cdot 1 = - 2.$$

Since the sequence $\left\{ S_n \right\}$ is bounded, the sequence $\left\{ W_n \right\}$ is unbounded, and the sequence $\left\{ T_n \right\}$ is unbounded, one can see that the condition that $\left\{ W_n \right\}$ be bounded is necessary, and lemma 2 is proved.

Lemma 3

A necessary and sufficient condition that the sequence $\left\{ T_n \right\}$ has a limit whenever the sequence $\left\{ S_n \right\}$ has the limit zero is that

(a) $\left\{ W_n \right\}$ be bounded, i.e. $W_n \leq W$ and,

(b) $\lim_{n \to \infty} a_{nv} = l_v$ exists for each v.

Then the limit of $T_n = \sum_{v=0}^{\infty} l_v S_v$.

Proof.

Consider the sufficiency first.

$$T_n = \sum_{v=0}^{\infty} a_{nv} S_v = \sum_{v=0}^{k} a_{nv} S_v + \sum_{v=k+1}^{\infty} a_{nv} S_v \quad \text{and}$$

$$\left|T_n - \sum_{v=0}^{k} a_{nv} S_v\right| \leq \sum_{v=k+1}^{\infty} |a_{uv}|\,|S_v| \leq L_k W, \text{ where}$$

$L_k = \text{l.u.b.}_{v \geq k} |S_v|$. Note that since the sequence $\{S_n\}$ has limit 0, L_k tends monotonically to 0. Now

$$\left|T_n - \sum_{v=0}^{k} a_{nv} S_v\right| \leq L_k W \text{ , so that}$$

$$- L_k W \leq T_n - \sum_{v=0}^{k} a_{nv} S_v \leq L_k W \text{ , so that}$$

$$\sum_{v=0}^{k} a_{nv} S_v - L_k W \leq T_n \leq \sum_{v=0}^{k} a_{nv} S_v + L_k W .$$

Let n approach infinity, and one has

$$\sum_{v=0}^{k} l_v S_v - L_k W \leq \underline{\lim}\, T_n \leq \overline{\lim}\, T_n \leq \sum_{v=0}^{k} l_v S_v + L_k W.$$

The sequence $\{T_n\}$ being bounded by lemma 2, these inequalities for k approaching infinity yield

$$\sum_{v=0}^{\infty} l_v\, S_v \leq \underline{\lim}\, T_n \leq \overline{\lim}\, T_n \leq \sum_{v=0}^{k} l_v\, S_v$$

since $L_k \rightarrow 0$. This proves the existence of $\lim_n T_n$, the convergence of the series $\sum_{v=0}^{\infty} l_v S_v$ and the equation $\lim_n T_n = \sum_{v=0}^{\infty} l_v\, S_v$.

To show the necessity of the conditions, a method similar to the one used in lemma 2 will be employed. To show that the existence of $\lim_n a_{nv}$ is a necessary condition, consider the set of sequences $S_n = \delta_{nv}$ (Kronecker delta) $n = 0,1,2,\ldots$. Then one has $T_n = a_{nv}$ and $\lim_n T_n$ exists by hypothesis. To show that the numbers W_n must be bounded, consider the set of integers defined in lemma 2, but now choose with the given k_n, y_n, u_n $n = 0,1,2,\ldots$

$$S_v = \begin{cases} \dfrac{\operatorname{sgn} a_{uv}}{\sqrt{W_u}} & k_n < v < y_n;\ u = u_n \\ 0 & \text{otherwise.} \end{cases}$$

S_v is defined in this manner because the sequence $\{S_n\}$ now has limit 0 (since the sequence $\{W_n\}$ is unbounded) and this is part of the hypothesis. The procedure is now exactly the same as in lemma 2 and for $u = u_n$

$$|T_u| \geq \frac{1}{3}\sqrt{W_n}\,,$$

which becomes infinite as $n \to \infty$, and the proof is complete.

Lemma 4

A necessary and sufficient condition that the sequence $\{T_n\}$ has the limit zero whenever the sequence $\{S_n\}$ has the limit zero is that

(a) the sequence $\{W_n\}$ is bounded and,

(b) $\lim_{n \to \infty} a_{nv} = 0 \qquad v = 0,1,2,3\cdots$

Proof.

The proof follows almost directly from lemma 3. From lemma 3 it is seen that

$$\lim T_n = \sum_{v=0}^{\infty} l_v S_v \text{ , where } \lim a_{nv} = l_v \text{ ;}$$

if $\lim a_{nv} = 0$ one has $\lim T_n = 0$. If $\lim T_n = 0$, then one can show $\lim a_{nv} = 0$ in the following manner. Consider the sequence

$S_i = (0,0,0,0,\ldots 0,1,0,\ldots)$ where the 1 is in the i th

position. Then $\lim S_i = 0$. This says $T_i = a_{ui}$, $u = u_n$ but $\lim T_n = 0 = l_v$. The rest of the proof follows from lemma 3.

Theorem 1

A necessary and sufficient condition that the sequence $\{T_n\}$ has a limit whenever the sequence $\{S_n\}$ has a limit is that

(a) $\{W_n\}$ is bounded,

(b) $\lim_{n \to \infty} a_{nv} = l_v \qquad v = 0,1,2,\ldots,$

(c) $\lim_{n \to \infty} V_n = \lim_{n \to \infty} \sum_{v=0}^{\infty} a_{nv}$ exists

Proof.

If one applies lemma 3 to the sequence $\{S_v - S\}$ where $S = \lim S_n$ then one has

$$\sum_{v=0}^{\infty} a_{nv}(S_v - S) = \lim_n (T_n - S\, V_n),$$

so that $$\lim_n T_n = S \lim V_n + \sum_{v=0}^{\infty} l_v(S_v - S)$$

and by condition (c) $\lim V_n$ exists, so that if the sequence $\{S_n\}$ has a limit, T_n also has a limit which is given by

$$\lim T_n = S \lim V_n.$$

SILVERMAN-TOEPLITZ THEOREM

Theorem 2

A necessary and sufficient condition that the sequence $\{T_n\}$ has a limit whenever the sequence $\{S_n\}$ has a limit and that the two limits are equal is that

(a) $\{W_n\}$ is bounded,

(b) $l_v = \lim_{n \to \infty} a_{nv} = 0, \quad v = 0,1,2,\ldots$

(c) $\lim_{n \to \infty} V_n = 1.$

Proof.

From Theorem 1, one can see that condition (a) is a necessary and sufficient to insure that $\{T_n\}$ has a limit whenever $\{S_n\}$ has a limit. Also from Theorem 1, recall that

$$\lim T_n = S \lim V_n + \sum_{v=0}^{\infty} lv\ (S_v - S).$$

If $lv = \lim_{n\to\infty} a_{nv} = 0$, and if $\lim V_n = 1$ then $\lim T_n = S$
If $\lim T_n = S$ then $\lim T_n = S = S \lim V_n$ which says $\lim V_n = 1$
The necessity of condition (b) follows from lemma 4, and the proof is complete.

APPLICATIONS

As the first application of this theorem, it will be shown that the (C,1) summability method is a regular method. Recall that the summability method was defined by

$$\lim_{n\to\infty} \frac{C_n}{n+1} \quad \text{where} \quad C_n = S_0 + S_1 + \dots S_n.$$

Therefore, one has as the infinite matrix of transformation and the transformed set of sequences the following:

$$
\left\| \begin{matrix} 1\ 0\ 0\ 0\ \cdots \\ \frac{1}{2}\ \frac{1}{2}\ 0\ 0\ 0\ \cdots \\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\ 0\ 0\ 0\ \cdots \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{n+1}\ \frac{1}{n+1}\ \frac{1}{n+1}\ \cdots \\ \cdot \\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{matrix} \right\| \left\| \begin{matrix} S_0 \\ S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ S_n \\ \cdot \\ \cdot \\ \cdot \end{matrix} \right\| = \left\| \begin{matrix} S_0 \\ \frac{S_0 + S_1}{2} \\ \frac{S_0 + S_1 + S_2}{3} \\ \\ \\ \\ \frac{S_0 + S_1 + \cdots S_n}{n+1} \\ \cdot \\ \cdot \\ \cdot \end{matrix} \right\|
$$

Now consider the first condition of the theorem: $\left\{ W_n \right\}$ is bounded. In this case

$$
W_n = \sum_{v=0}^{\infty} \left| a_{nv} \right| = (n+1)\left(\frac{1}{n+1}\right) = 1
$$

and condition (a) is obviously satisfied. Condition (b) is:

$l_v = \lim_{n \to} a_{nv} = 0$; $v = 0,1,2,\cdots$; for an arbitrary v one has

$a_{nv} = \frac{1}{n+1}$ and $\lim_{n \to \infty} \frac{1}{n+1} = 0$. Condition (c) is: $\lim_{n \to \infty} V_n = 1$.

In this case one has

$$
\lim V_n = \lim \sum_{v=0}^{\infty} a_{nv} = \lim \sum_{v=1}^{n} \frac{1}{n} = \lim n \left(\frac{1}{n}\right) = 1.
$$

From the fact that all three conditions of the Silverman-Toeplitz Theorem are satisfied, one knows that (C,1) summability is a regular method.

As a second example, consider the Euler method of summability. Recall that the method is defined for a fixed complex number r. Using the Silverman-Toeplitz Theorem, the values of r for which this method is regular shall now be determined. The Euler summability method is given by:

$$T_n = \sum_{k=0}^{n} a_{nk} s_k ,$$

where $$(\Theta r + (1-r))^n = \sum_{k=1}^{\infty} a_{nk} \Theta^k ,$$

so that $$a_{nk} = \binom{n}{k} r^k (1-r)^{n-k}.$$

The infinite matrix of transformation is given by:

$$\left\| \begin{matrix} 1 & 0 & 0 & \cdots & & \\ 1-r & r & 0 & \cdots & & \\ (1-r)^2 & 2r(1-r) & r^2 & 0 \cdots & & \\ \vdots & & & & & \\ (1-r)^n & \binom{n}{1} r(1-r)^{n-1} & \cdots & \binom{n}{s} r^s(1-r)^{n-s} & 0 & \cdots \\ \vdots & & & & & \end{matrix} \right\|$$

To determine the set of r's for which the Euler method is regular, assume that the first condition of the Silverman-Toeplitz Theorem is satisfied, that is $\{W_n\}$ is bounded. From this it is seen that $\sum_{v=0}^{\infty} |a_{nv}|$ is bounded. For the Euler transformation, one sees that $\sum_{v=0}^{m} |a_{nv}| = (|r| + |1-r|)^m$, $n = 0,1,2,\ldots$.

In the case where r is an arbitrary complex constant, let $r = a + bi$ and consider $r = Re^{i\theta}$ where $R = \sqrt{a^2 + b^2}$ and $\theta = \arctan \frac{b}{a}$. Then

$$(|r| + |1-r|) = (R + |1 - R(\cos\theta + i\sin\theta)|)$$

$$= (R + \sqrt{1 - 2R\cos\theta + R^2})$$

$$= (R + \sqrt{1 - 2R + R^2 + 2R - 2R\cos\theta})$$

$$= R + (1-R)\sqrt{1 + \frac{2R(1-\cos\theta)}{(1-R)^2}} \; .$$

Since $1 - \cos\theta \geq 0$, $\sqrt{1 + \frac{2R(1-\cos\theta)}{(1-R)^2}} \geq 1$, so that

$$(|r| + |1-r|) \geq R + (1-R) \cdot 1 = 1,$$

and equals 1 if and only if $\theta = 2k\pi$; $k = 0,1,2,\ldots$. If $\theta \neq 2k\pi$, then $(|r| + |1-r|) \geq 1$, and one can readily see that condition (a) is not satisfied since

$$(|r| + |1-r|)^n \to \infty \quad \text{as} \quad n \to \infty .$$

Therefore, if condition (a) of the theorem is to be satisfied, one must have $\theta = 2k\pi$, and this says that r must be real. If r is real then condition (a) is satisfied if and only if $0 \leq r \leq 1$.

After consideration of the first condition, r has been restricted to be real and in the interval $0 \leq r \leq 1$, if the Euler method of summability is to be regular. The two cases where $r = 0$ and $r = 1$ will be considered separately. For these values, the infinite matrices of transformation are given by:

$$\left\| \begin{matrix} 1\ 0\ 0 \cdots \\ 1\ 0\ 0 \cdots \\ \cdot \\ \cdot \\ \cdot \\ 1\ 0\ 0 \cdots \\ \cdot \\ \cdot \\ \cdot \ \cdots\ \cdots \end{matrix} \right\| \left\| \begin{matrix} s_0 \\ s_1 \\ \cdot \\ \cdot \\ \cdot \\ s_n \\ \cdot \\ \cdot \\ \cdot \end{matrix} \right\| = \left\| \begin{matrix} s_0 \\ s_0 \\ \cdot \\ \cdot \\ \cdot \\ s_0 \\ \cdot \\ \cdot \\ \cdot \end{matrix} \right\|$$

$$\left\| \begin{matrix} 1\ 0\ 0 \\ 0\ 1\ 0\ 0 \\ 0\ 0\ 1\ 0\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0\ 0 \cdots 0\ 1\ 0\ \cdot \\ \cdot \\ \cdot \\ \cdot \ \cdots\ \cdots \end{matrix} \right\| \left\| \begin{matrix} s_0 \\ s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_n \\ \cdot \\ \cdot \\ \cdot \end{matrix} \right\| \quad \left\| \begin{matrix} s_0 \\ s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_n \\ \cdot \\ \cdot \\ \cdot \end{matrix} \right\|$$

The first set of matrices are those obtained for $r = 0$, and the second for the case where $r = 1$. In the first case, condition (b) of the Silverman-Toeplitz Theorem is not satisfied for $v = 0$, and therefore when $r = 0$, the Euler method is not regular. In the second case where $r = 1$, one can see that this is just the identity transformation, which is obviously regular.

For the last two conditions of the theorem, one needs only to consider r real and in the interval $0 < r < 1$. Consider condition (b) $\lim_{n \to \infty} a_{nv} = 0$, for $v = 0,1,2,\ldots$. To show that this condition is satisfied, consider the infinite series whose nth term is the a_{nv} of condition (b). It will be shown that the series converges and therefore that the nth term approaches 0, which in turn will show that $\lim_{n \to \infty} a_{nv} = 0$. Consider the infinite series

$$\sum_{n=0}^{\infty} \binom{n+v}{v} r^v \quad (1-r)^{n-v},$$

where v is fixed. Now apply the ratio test to this, and one has

$$\lim_{n \to \infty} \left| \frac{\binom{n+1+k}{k} (1-r)^{n+1-k}}{\binom{n+k}{k} (1-r)^{n-k}} \right| = \lim_{n \to \infty} \left| 1 + \frac{k}{n+1} \right| \left| 1-r \right|$$

$$= \left| 1-r \right| .$$

For $0 < r < 1$, this series converges and therefore the nth term approaches 0 and $\lim_{n \to \infty} a_{nv} = 0$.

Condition (c) is obviously satisfied since it has been shown that

$$\sum_{v=0}^{n} \left| a_{nv} \right| = (|r| + |1-r|)^n,$$

but since $r > 0$ this is $(r + 1-r)^n = (1)^n = 1$. The conclusions that one can draw from the Silverman-Toeplitz Theorem concerning the Euler method of summability is that the method is regular if and only if r is real and in the interval $0 < r \leq 1$.

From these two examples, the importance of the Silverman-Toeplitz Theorem as a tool in the theory of divergent series is evident. It can be used to determine whether a given summability method is regular. In certain cases, it can be used to determine the values for which a method is regular. In either case, the Theorem gives much information and also leads to many interesting mathematical problems.

ACKNOWLEDGEMENT

The writer wishes to express his sincere appreciation to Dr. C. F. Koch for introducing him to this subject and for his patient guidance and supervision given during the preparation of this report.

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THE SILVERMAN-TOEPLITZ THEOREM

by

BRIAN RUDER

B. A., Fort Hays Kansas State College, 1964

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1966

The modern theory of summability methods and divergent series had its start in the eighteenth century. Some of the first men to work in this area were Leibnitz and Euler.

In this paper methods for assigning values to certain divergent series are discussed. If these methods are to be of any value, certain restrictions must be placed on them. One of the more important conditions that the method must possess is called regularity. To say that a summability method is regular means that the method "sums" convergent series to the value that the series converges to in its circle of convergence. The Silverman-Toeplitz Theorem supplies three necessary and sufficient conditions that a method must satisfy if it is to be regular. A proof of the Silverman-Toeplitz Theorem is given in this paper. After the Theorem is proved, two applications of the Theorem are given.